On the approximated analysis of the orbit evolution of artificial satellites

By

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Methods of classic celestial mechanics for an approximated study of the limited three body problem by averaging the disturbing accelerations [1] acquire great significance for the missions* connected with artificial satellites. Both the qualitative picture of the evolution, which can be obtained from averaged equations, and approximated quantitative estimates based on simplified formulas are very useful for such purposes.

At present the evolution of individual orbits of artificial satellites at a considerable time interval can be calculated on computers by numerical integration of the precise set of differential equations. Comparison of these solutions with the results obtained from approximated formulas is a sufficiently reliable practical estimate of the accuracy of approximated equations. Such comparison (alongside the estimates of the orders of magnitude of ignored values) permits to conclude about the applicability of the obtained approximated equations for considering the evolution of some classes of the orbits.

For many practical problems connected with artificial satellites element variation estimates are required at a limited time interval with accuracies relatively not high. This enables us to use approximated equations for a broad class of artificial satellites orbits. The latter in its turn stimulates a detailed study of approximated averaged equations.

In the present paper we shall not consider the more precise averaging scheme with respect to one or several revolutions which allowed us to obtain formulas for calculating the element variation depending on the satellite successive revolution and formulas determining the orbital elements change during the time of one revolution of the disturbing body. This problem was analysed in the author's paper [2]. If element variations during the time of one revolution of the disturbing body turn to be sufficiently small, a scheme is applicable (in any case for qualitative estimates) of independent twofold averaging of disturbing accelerations with respect to the mean motion of the satellite and the mean motion of the disturbing body. In the theory of a first approximation the satellite orbital elements variation at an averaging interval is not taken into account when equations of evolution are obtained.

1. The formulation of the problem

The problem of the satellite orbit evolution is considered in the following formulation:

(a) The satellite rotates around the central body which has the axisymmetric gravitational potential V determined by the formula (3)

$$\left.\begin{aligned} V &= \frac{fM}{r} - \frac{fM\delta}{3r^3}(3\,\mathrm{Sin}^2\psi - 1), \\ \delta &= R^2\left(\alpha - \frac{m}{2}\right); \qquad m = \frac{\Omega^2 R}{G_R}; \end{aligned}\right\} \qquad (1)$$

f is the gravitational constant.
M is the mass of the central body.
ψ is the latitude of the point calculated from the plane of gravitational symmetry. We shall call this plane the equatorial plane.
R is the equatorial radius of the central body.
α is the oblateness of the central body.
Ω is the angular velocity of the body own rotation.
G_R is the gravity acceleration at the equator.

(b) The satellite expperiences the disturbances of the outer gravitating point with mass M_b which rotates along an elliptical orbit with small eccentricity e_b and period T_b.

(c) The maximum distance of the satellite from the central body $r_{\max}$ is much lower than the value of semiaxis a_b of the disturbing body orbit. Disturbing accelerations are considered to a first approximation with respect to the value $\frac{r_{\max}}{a_b} \ll 1$.

This formulation of the problem after twofold averaging leads to the following system of evolution equations:

$$\left.\begin{aligned} \frac{da}{dn} &= 0, \\ \frac{d\varepsilon}{dn} &= -(1-\varepsilon)\,\varepsilon^{1/2}\,\mathrm{Sin}^2 i\,\mathrm{Sin}\,2\omega, \\ \frac{di}{dn} &= -\frac{1}{2}\,\frac{(1-\varepsilon)}{\varepsilon^{1/2}}\,\mathrm{Sin}\,i\,\mathrm{Cos}\,i\,\mathrm{Sin}\,2\omega + \beta\,\frac{\mathrm{Sin}\,I\,\mathrm{Sin}\,\Omega}{\varepsilon^2}\,\mathrm{Cos}\,i_{\mathrm{eq}}, \\ \frac{d\Omega}{dn} &= -\frac{\mathrm{Cos}\,i}{\varepsilon^{1/2}}\left[(1-\varepsilon)\,\mathrm{Sin}^2\omega + \frac{\varepsilon}{5}\right] + \\ &\quad + \beta\,\frac{\mathrm{Cos}\,i\,\mathrm{Sin}\,I\,\mathrm{Cos}\,\Omega - \mathrm{Sin}\,i\,\mathrm{Cos}\,I}{\mathrm{Sin}\,i\,\varepsilon^2}\,\mathrm{Cos}\,i_{\mathrm{eq}}, \\ \frac{d\omega}{dn} &= \frac{(\mathrm{Cos}^2 i - \varepsilon)\,\mathrm{Sin}^2\omega + \frac{2}{5}\varepsilon}{\varepsilon^{1/2}} + \\ &\quad + \frac{1}{2}\,\frac{\beta}{\varepsilon^2}\left(5\,\mathrm{Cos}^2 i_{\mathrm{eq}} - 1 - \frac{2\,\mathrm{Sin}\,I\,\mathrm{Cos}\,\Omega}{\mathrm{Sin}\,i}\,\mathrm{Cos}\,i_{\mathrm{eq}}\right). \end{aligned}\right\} \qquad (2)$$

$$\left.\begin{aligned}
&\mathrm{Cos}\, i_{\mathrm{eq}} = \mathrm{Cos}\, i\, \mathrm{Cos}\, I + \mathrm{Sin}\, i\, \mathrm{Sin}\, I\, \mathrm{Cos}\, \Omega\,, \\
&n = A\,N\,, \qquad A = \frac{15}{2}\,\pi\,\frac{M_b}{M}\left(\frac{a}{a_b}\right)^3 \varepsilon_b^{-\frac{3}{2}}, \\
&\beta = \frac{4}{15}\,\frac{\delta\, a_b^3\, \varepsilon_b^{3/2}}{a^5}\left(\frac{M}{M_b}\right), \qquad \varepsilon = 1 - e^2, \qquad \varepsilon_b = 1 - e_b^2.
\end{aligned}\right\} \quad (3)$$

Here M is mass of the central body, a, e, i, Ω, ω are generally accepted designations for artificial satellite orbital elements. Angular elements are calculated from the disturbing body orbit plane. The node position Ω is calculated from the descending node of the disturbing body orbit in the plane of the equator.

I is the inclination of the equatorial plane to the plane of the disturbing body orbit.

N is the order number of satellite revolutions.

Besides the trivial integral $a = \mathrm{const}$ the considered system has the integral [1]

$$\overline{w} = \mathrm{const},$$

where $\overline{w}$ is the twice averaged total potential of disturbing accelerations of the outer attracting point and of noncentricity of the main field.

In the considered approximation this integral takes the form:

$$(1-\varepsilon)\left(\frac{2}{5} - \mathrm{Sin}^2\omega\, \mathrm{Sin}^2 i\right) + \frac{\varepsilon\, \mathrm{Cos}^2 i}{5} + \frac{\beta}{\varepsilon^{3/2}}\left(\mathrm{Cos}^2 i_{\mathrm{eq}} - \frac{1}{3}\right) = \mathrm{const}. \quad (4)$$

2. The disturbance of the gravitating point

In the case when $\beta = 0$ the set of equations has three integrals:

$$\left.\begin{aligned}
&a = \mathrm{const}, \\
&\varepsilon\, \mathrm{Cos}^2 i = c_1, \\
&(1-\varepsilon)\left(\frac{2}{5} - \mathrm{Sin}^2\omega\, \mathrm{Sin}^2 i\right) = c_2.
\end{aligned}\right\} \quad (5)$$

The range of possible values of constants c_1 and c_2 is represented in Figure 1.

In the case $c_2 > 0$ the argument of the latitude of the pericentre (angular distance of the pericentre from the node) ω monotonously increases: $\delta\omega > 0$. ε and i oscillate.

Maximum $\varepsilon_{\max}$ and minimum $\varepsilon_{\min}$ are realized at $\mathrm{Sin}\, 2\omega = 0$ and determined by formulas:

$$\left.\begin{aligned}
&\varepsilon_{\max} = 1 - \frac{5}{2}\,c_2, \\
&\varepsilon_{\min} = \frac{1}{2}\left[1 + \frac{5}{3}(c_1 + c_2) - \sqrt{\left[1 + \frac{5}{3}(c_1+c_2)\right]^2 - \frac{5}{3}\cdot 4c_1}\,\right].
\end{aligned}\right\} \quad (6)$$

The corresponding extremal values for the inclination are found from the second integral (5) in the form

$$(\mathrm{Cos}^2 i)\,\min = \frac{c_1}{\varepsilon_{\max}}\,; \qquad (\mathrm{Cos}^2 i)\,\max = \frac{c_1}{\varepsilon_{\min}}\,.$$

In the case $c_2 < 0$ the argument of the latitude of the perigee ω also varies within limited bounds. Extremal values ω_{extrem} are found from the relationships:

$$\mathrm{Sin}^2\omega_{\mathrm{extrem}} = \frac{\frac{2}{5}}{1 - \frac{c_1}{\varepsilon^{*2}}}\,, \quad (7)$$

where ε^* is determined from the equation

$$-\left(c_2 + \frac{2}{5}\,c_1\right)(\varepsilon^*)^2 + \frac{4}{5}\,c_1\,\varepsilon^* - c_1\left(\frac{2}{5} - c_2\right) = 0 \quad (8)$$

and the condition $0 \leqq \varepsilon^* \leqq 1$.

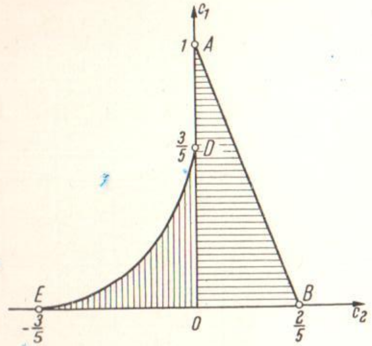

Fig. 1. The region of possible values of constants c_1 and c_2.

Two values $\omega_{\min}$ and $\omega_{\max}$ are obtained as two roots of Eq. (7) which are symmetrical with respect to $\pi/2$ [if $\omega_0 \subset (\omega_1^*, \omega_3^*)$] or $\frac{3}{2}\pi$ [if initial $\omega_0 \subset (\omega_1^*, 2\pi + \omega_2^*)$]. In these formulas and subsequently the following designations for special values of ω are introduced:

$$\left.\begin{aligned}
&\omega_1^* = \frac{1}{2}\,\mathrm{arc\,Cos}\,\frac{1}{5}\,; \qquad \omega_2^* = -\omega_1^*\,; \\
&\omega_3^* = \pi - \omega_1^*\,; \qquad \omega_4^* = \pi + \omega_1^*\,.
\end{aligned}\right\} \quad (9)$$

In the case $c_2 < 0$ extremal values $\varepsilon_{\mathrm{extrem}}$ are two roots of a quadratic equation:

$$\varepsilon^2_{\mathrm{extrem}} - \left[1 + \frac{5}{3}(c_1 + c_2)\right]\varepsilon_{\mathrm{extrem}} + \frac{5}{3}\,c_1 = 0\,. \quad (10)$$

Dependences of extremal values ε and ω from constants c_1 and c_2 are represented in Figures 2, 3, 4.

If $e_{\max} = \sqrt{1 - \varepsilon_{\min}} > 1 - \frac{R}{a}$ the fall of the satellite onto the central body takes place.

The limit of the range of permissible values of constants c_1 and c_2 corresponds to a number of limit cases:

1. Line AB $c_2 = \frac{2}{5}(1 - c_1)$ corresponds to the case when the initial inclination i_0 is equal to 0 or 180°.

In this case $\varepsilon = \varepsilon_0$, the latitude of the pericentre φ which is calculated from the disturbing body orbit plane also does not vary and is equal to zero.

The longitude of the pericentre α changes with velocity

$$\frac{d\alpha}{dn} = \pm \frac{1}{5} \varepsilon_0^{1/2} \tag{11}$$

(sign plus corresponds to $i_0 = 0$, sign minus corresponds to $-i_0 = 180°$).

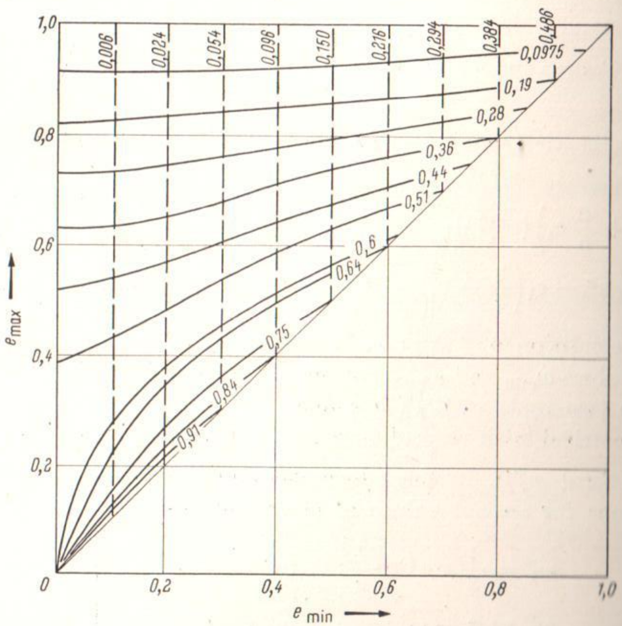

Fig. 2. Extremal values of the eccentricity depending on c_1 (solid lines) and $c_2 > 0$ (dashed lines). Numbers near the lines denote constants c_1 and $1.5c_2$.

2. Line ED $c_2 = -\frac{3}{5}\left(1 - \sqrt{\frac{5}{3} c_1}\right)^2$ corresponds to special initial data

$$\mathrm{Sin}^2 \omega_0 = 1, \qquad \mathrm{Cos}^2 i_0 = \frac{3}{5} \varepsilon_0 .$$

In this case $\varepsilon = \varepsilon_0$, $i = i_0$, $\varphi = \varphi_0$ and the evolution will represent only the turn of the orbit about the normal to the plane of the orbit of the disturbing body with a velocity

$$\frac{d\alpha}{dn} = \pm \frac{1}{5} \sqrt{\frac{5}{3}} \left(\frac{12}{3} \varepsilon_0 - 3\right) \tag{12}$$

("+" if $\mathrm{Cos}\, i_0 > 0$, "−" if $\mathrm{Cos}\, i_0 < 0$).

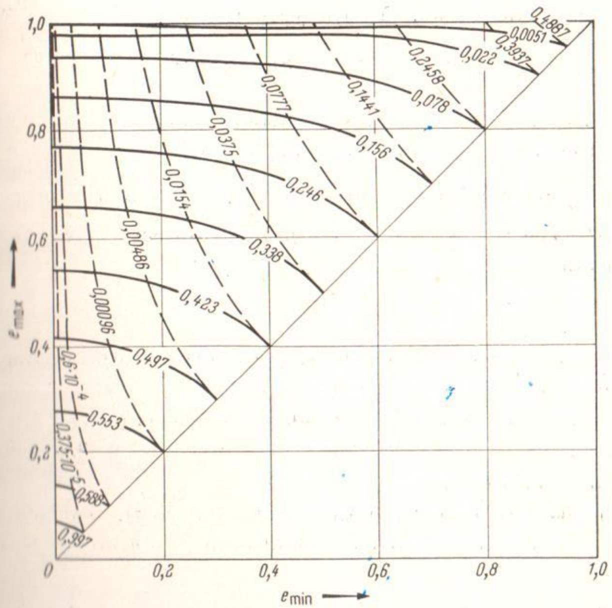

Fig. 3. Extremal values of the eccentricity depending on c_1 (solid lines) and $c_2' < 0$ (dashed lines). Numbers near the lines denote constants c_1 and c_2.

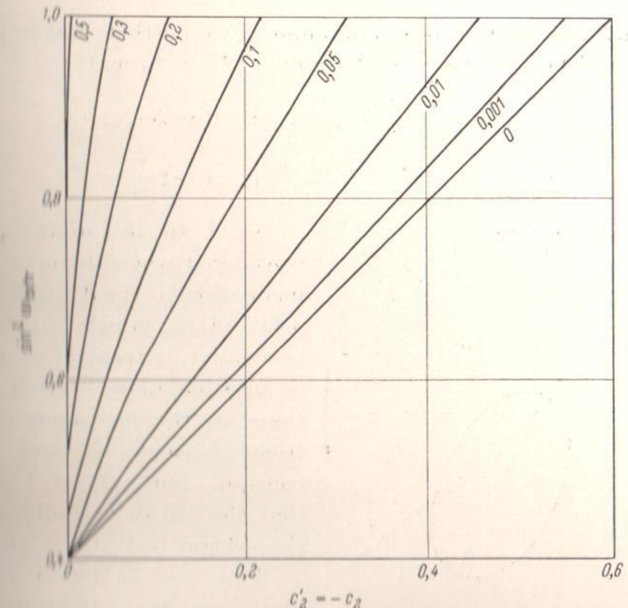

Fig. 4.
Extremal values of $\sin^2 \omega$ depending on $c_2 < 0$ at different values of c_1 (numbers near the lines).

3. Line OA corresponds to the case when

$$\operatorname{Sin}^2 \varphi_0 = \operatorname{Sin}^2 i_0 \operatorname{Sin}^2 \omega_0 = \frac{2}{5}.$$

In this case $\omega \to \omega_3^*$ [if $\omega_0 \subset (\omega_3^*, \omega_1^*)$] or $\omega \to \omega_2$ [if $\omega_0 \subset (\omega_4^*, 2\pi + \omega_2^*)$]. The orbit tends to a circular one, i.e. $e \to 0$.

If $\omega_0 < \frac{\pi}{2}$ (or correspondingly $\omega_0 < \frac{3}{2}\pi$) then in the process of the evolution the eccentricity local maximum is observed. It is determined by the formula

$$e_{\max} = \sqrt{1 - \frac{5}{3}\varepsilon_0 \operatorname{Cos}^3 i_0}\,. \qquad (13)$$

4. Line BE corresponds to the case when the initial inclination i_0 is equal to $90°$. In this case $i = 90° = \text{const.}$

In the process of evolution ω tends to special values ω_1^* [in the case, if $\omega_0 \subset (\omega_2^*, \omega_3^*)$] or $\omega \to \omega_4^*$ [if $\omega_0 \subset (\omega_3^*, 2\pi + \omega_2^*)$]. The eccentricity of the orbit tends to unity.

For a central body with radius R the fall to the surface will take place at $\varepsilon = \tilde{\varepsilon}$ and $\omega = \tilde{\omega}$ where $\tilde{\varepsilon}$ and $\tilde{\omega}$ are determined by formulas

$$\left.\begin{aligned} \tilde{\varepsilon} &= 1 - \left(1 - \frac{R}{a}\right)^2, \\ 5\operatorname{Cos}2\tilde{\omega} - 1 &= \frac{(1-\varepsilon_0)(5\operatorname{Cos}2\omega_0 - 1)}{1-\tilde{\varepsilon}}\,. \end{aligned}\right\} \qquad (14)$$

If $(\sin 2\omega_0) < 0$ the local minimum of eccentricity takes place at $\sin 2\omega = 0$. The $e_{\min}$ value is determined from formulas

$$e_{\min}^2 = \frac{e_0^2(5\operatorname{Cos}2\omega_0 - 1)}{4}, \quad \text{if} \quad \operatorname{Cos}2\omega_0 > \frac{1}{5},$$

$$e_{\min}^2 = \frac{e_0^2(1 - 5\operatorname{Cos}2\omega_0)}{6}, \quad \text{if} \quad \operatorname{Cos}2\omega_0 < \frac{1}{5}.$$

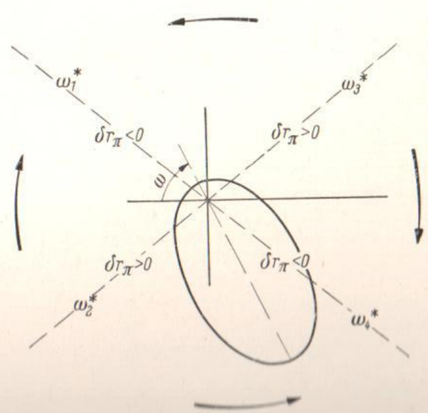

Fig. 5. Evolution of the orbit at $i = 90°$.

Thus, though in the problem considered when the orbital plane inclination to the disturbing body orbit plane equals to $90°$ the temporary decrease of eccentricity is possible under certain conditions, in the last analysis eccentricity becomes so large (at the constant semiaxis of the orbit) that the fall of the satellite onto the central body takes place.

This case is interesting as a limit one for sufficiently large inclinations.

In the vicinity of $i_0 = 90°$ there is a continuous dependence of $e_{\max}$ on i_0.

In Figure 5 the orbit evolution for the case $i_0 = 90°$ is illustrated. The sign of the change of the pericentre height is marked in the quadrants of the coordinate system. Besides, dashed lines are shown which denote special values of the argument of the latitude of the perigee $\omega_1^*, \omega_2^*, \omega_3^*, \omega_4^*$. The arrows between dashed lines indicate the direction of the ω change depending on the perigee position.

The dependence of ω on the time t for the orbit with the inclination equal to $90°$ is determined from the formulas

$$\left.\begin{aligned} t &= \frac{2\pi}{\sqrt{fM}} a^{3/2} N, \\ N &= \frac{a_N}{A}[\delta F(\varphi, \varkappa^2) - \delta_0 F(\varphi_0, \varkappa^2)], \end{aligned}\right\} \qquad (15)$$

where A is according to (3), F is an elliptical integral of the first kind

$$\delta = 1 \quad \text{if} \quad \operatorname{Sin}2\omega > 0; \qquad \delta = -1 \quad \text{if} \quad \operatorname{Sin}2\omega < 0,$$

$$\delta_0 = 1 \quad \text{if} \quad \operatorname{Sin}2\omega_0 > 0; \qquad \delta_0 = -1 \quad \text{if} \quad \operatorname{Sin}2\omega_0 < 0.$$

Values φ_0, φ, $\varkappa^2$ and a_N are determined from different formulas depending on the sign of the integral

$$c = e_0^2(5\operatorname{Cos}2\omega_0 - 1).$$

	$c>0$	$c<0$
B	$\frac{c+1}{5}$	$\frac{c+1}{5}$
a_N	$2\left[\frac{4}{5}(1+B)\right]^{-\frac{1}{2}}$	$2\left[\frac{6}{5}(1-B)\right]^{-\frac{1}{2}}$
$\operatorname{Sin}\varphi_0$	$\left[\frac{(1+B)(1-\operatorname{Cos}2\omega_0)}{(1-B)(1+\operatorname{Cos}2\omega_0)}\right]^{\frac{1}{2}}$	$\left[\frac{(1-B)(1+\operatorname{Cos}2\omega_0)}{(1+B)(1-\operatorname{Cos}2\omega_0)}\right]^{\frac{1}{2}}$
$\operatorname{Sin}\varphi$	$\left[\frac{(1+B)(1-\operatorname{Cos}2\omega)}{(1-B)(1+\operatorname{Cos}2\omega)}\right]^{\frac{1}{2}}$	$\left[\frac{(1-B)(1+\operatorname{Cos}2\omega)}{(1+B)(1-\operatorname{Cos}2\omega)}\right]^{\frac{1}{2}}$
$\varkappa^2$	$\frac{3}{2}\,\frac{(1-B)}{(1+B)}$	$\frac{2}{3}\,\frac{(1+B)}{(1-B)}$

The lifetime of the satellite as a function of the initial data is determined from formula (15) at $\omega = \tilde{\omega}$ where $\tilde{\omega}$ is according to (14).

In cases when the initial value $\omega_0 = \omega_1^*$ or $\omega = \omega_4^*$, the satellite lifetime is calculated according to the elementary formula:

$$t = \frac{2\pi}{\sqrt{fM}} a^{3/2} N, \qquad (16)$$

where
$$N = \frac{5}{\sqrt{24}}\,\frac{1}{A}\ln\left[\frac{(1+\sqrt{1-e_0^2})\,(1-\sqrt{1-\tilde{e}^2})}{(1-\sqrt{1-e_0^2})\,(1+\sqrt{1-\tilde{e}^2})}\right],$$
$$\tilde{e} = 1 - \frac{R}{a}.$$

3. The influence of the noncentricity of the gravitational field

The correctness of the conclusion about the fall of the satellite onto the central body (see the previous section) at the satellite inclination to the disturbing body orbit plane close to 90° was checked by numerical integration of the precise set of equations for several orbits of artificial satellites of the Moon.

However, practical validity of this approximated conclusion, generally speaking, can be doubted by the fact of the existence of satellites of Uranus. Satellites of Uranus rotate about the equator of the planet. The inclination of the equator of Uranus to its orbit is 98°, i.e. orbital planes of satellites of Uranus are inclined to the orbital plane of the disturbing body — the Sun — at an angle close to 90°.

We shall show that the stationary eccentricity of these satellites orbits within the framework of the system of Eq. (2) can be explained if disturbances at the expense of the Uranus gravitational field oblateness are taken into account.

Somewhat idealizing the problem let us assume that in system (2) $I = 90°$, $i_0 = 90°$. In this case the set of equations (2) is transformed as follows
$$\left.\begin{aligned} &\frac{da}{dn} = 0; \quad \frac{di}{dn}; \quad \frac{d\Omega}{dn} = 0,\\ &\frac{d\varepsilon}{dn} = -(1-\varepsilon)\,\varepsilon^{1/2}\,\mathrm{Sin}2\omega,\\ &\frac{d\omega}{dn} = \varepsilon^{1/2}\left(\frac{2}{5} - \mathrm{Sin}^2\omega\right) + \frac{\beta}{\varepsilon^2}.\end{aligned}\right\} \tag{17}$$

After similar substitution integral (4) is written in the form
$$(1-\varepsilon)\left(\frac{2}{5} - \mathrm{Sin}^2\omega\right) + \frac{2}{3}\,\frac{\beta}{\varepsilon^{3/2}} = c. \tag{18}$$

The charakteristic behaviour of integral curves in the plane (ω, ε) for the case $0 < \beta < \frac{3}{5}$ is illustrated in Figure 6.

Arrows show the direction of the evolution. In the figure dashed curves are indicated: the vertical tangents curve AOB whose equation is of the following form
$$\varepsilon = \left(\frac{\beta}{\mathrm{Sin}^2\omega - \frac{2}{5}}\right)^{2/5} \tag{19}$$
and special values for the case $\beta = 0$, $\omega = \omega_1^*$ and $\omega = \omega_3^*$.

Point 0 represents a stationary solution
$$\varepsilon = \varepsilon_0 = \left(\frac{5}{3}\beta\right)^{2/5}, \qquad \omega = \omega_0 = \frac{\pi}{2}.$$

If $\beta \to 0$, curve (19) tends to fill the whole rectangle which is limited by dashed vertical lines, and the minimum value ε tends to zero, i.e. the value of the eccentricity maximum in the process of evolution approaches unity, and the fall of the satellite onto the central body is realized.

If $\beta > \frac{3}{5}$, there will be no libration point 0 in the plane (ε, ω).

In the case $\beta \neq 0$ maximum and minimum values of the eccentricity are determined from the equations:

At a) $\beta > \frac{3}{5}$ and b) $\beta < \frac{3}{5}$, $c > \frac{2}{3}\beta$,
$$\left.\begin{aligned} &\frac{2}{5}e_{\min}^2 + \frac{2}{3}\,\frac{\beta}{(1-e_{\min}^2)^{3/2}} = c,\\ &-\frac{3}{5}e_{\max}^2 + \frac{2}{3}\,\frac{\beta}{(1-e_{\max}^2)^{3/2}} = c.\end{aligned}\right\} \tag{20}$$

At c) $\beta < \frac{3}{5}$, $c < \frac{2}{3}\beta$ the extremal values of the eccentricity are found as two roots of the equation
$$-\frac{3}{5}e_{\mathrm{extr}}^2 + \frac{2}{3}\,\frac{\beta}{(1-e_{\mathrm{extr}}^2)^{3/2}} = c. \tag{21}$$

For Oberon, a satellite of Uranus, which is the farthest from the planet and wich is maximally disturbed by the Sun, it is possible to obtain an approximated estimate for the value β.

Having supposed that the gravitational potential of the field of Uranus can be approximated by formula (1) we obtain after the substitution of approximated values of all the parameters:
$$\beta \approx 91.$$

For large values β and small values of the eccentricity the amplitude of oscillations according to (20) will be approximately as follows:
$$e_{\max} - e_{\min} \approx \frac{1}{2\beta}\,e_{\min}$$

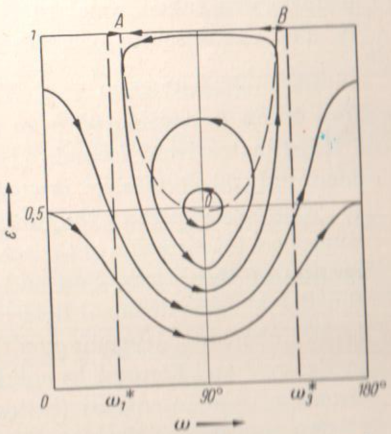

Fig. 6. The typical field of integral curves in the plane (ε, ω) at $i = 90°$, $I = 90°$, $\beta < \frac{3}{5}$.

i.e. for the satellite of Uranus ($e \approx 0.001$) the long-period oscillations of the eccentricity can be of the order of 10^{-5}.

The considerable change of the character of evolution, when the noncentricity of the main field is taken into account, is not typical of satellites of the solar system planets.

As one more example where the oblateness does not influence the qualitative conclusions obtained from equations at $\beta = 0$ we consider the

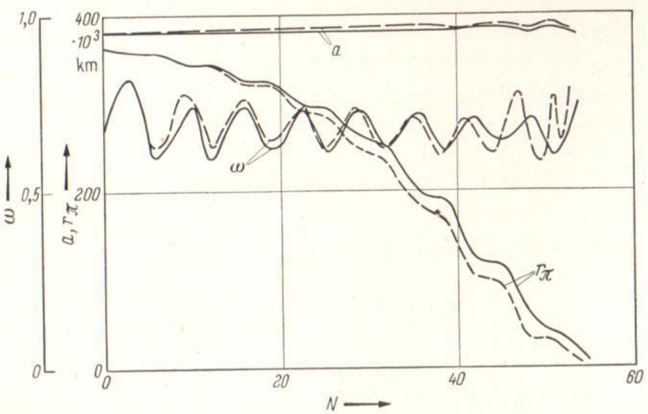

Fig. 7. The orbital element variation a, r, ω in a precise calculation (solid lines) and in an approximated calculation according to the method of single averaging (dashed lines).

orbit evolution of an Earth satellite, whose semiaxis and eccentricity are equal to the semiaxis and the eccentricity of the Moon's orbit, respectively.

The satellite orbital inclination with respect to the plane of the ecliptic was taken equal to 90°.

For the satellite considered the parameter $\beta = 0.2 \times 10^{-4}$. If the initial value is $\omega_0 = \omega_1^* = \frac{1}{2} \text{arc cos} \frac{1}{5}$, the lifetime of such satellite at $\beta = 0$ can be estimated from formula (16).

Estimates from formula (16) have shown that such satellite could made only 52 circuits, i.e. it would exist on the orbit for about four years.

Exact estimates of the lifetime of such a "Moon" were obtained by numerical intergration of the set of differential equations of the combined motion of the Earth, Sun and the satellite with the noncentricity of the Earth's gravitational field taken into account. The evolution of the orbital elements depending on the order number of circuits is presented in Figure 7 and Figure 8 by solid lines. The minimum distance at a given circuit r_π was calculated (instead of the orbit eccentricity).

For comparison the same dependences obtained by means of approximated differential equations averaged once with respect to the satellite revolution [2] are indicated by dashed lines.

In a precise solution the minimum distance of the satellite orbit becomes less than the Earth radius after 55 revolutions. The lifetime was 54 revolutions when calculations were made according to the equations averaged once.

This example illustrates the efficiency of the approximated formulas not only in a qualitative, but also in a quantitative aspect.

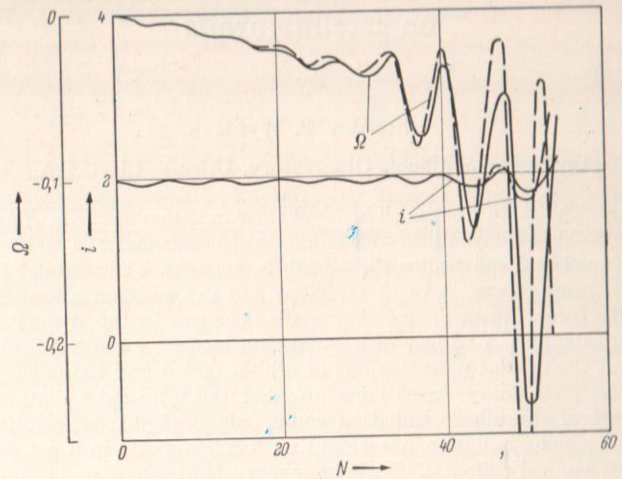

Fig. 8. The orbital element variation i, Ω in a precise calculation (solid lines) and in an approximated calculation according to the method of single averaging.

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[2] Lidov, M. L.: Evolution of Orbits of Planetary Artificial Satellites Under the Influence of Gravitational Disturbances of Outer Bodies. Artificial Earth Satellites, The U.S.S.R. Academy of Sciences Publishing House, vol. 8, 1961.

Discussion

During the discussion of the above paper, Dr. I. I. Shapiro did express the following remarks:

The work of Drs. Musen and Lidov may have important cosmological implications: the solar system in the past may have more closely resembled a three-dimensional complex of orbiting bodies. Perhaps formerly existing, unstable satellites could provide explanations for such anomalies as the tilted axis of the earth's rotation, craters on the moon, and the so-called "missing" planet. The last may have existed and have been destroyed by interaction with a large satellite in a highly inclined orbit. Similarly, a former satellite of the earth may have imparted sufficient angular momentum to cause rotation about an inclined axis. It may also be possible that some craters on the moon were created by impacts with the remains of an unstable lunar satellite, broken up after passing within Roche's limit.